

# GROMOV WITTEN INVARIANTS FOR MAXIMAL PARABOLIC VECTOR BUNDLES OVER AN ORBIFOLD

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**ABSTRACT.** We define the Gromov-Witten invariants for the parabolic bundles over an orbifold  $C$  in various situation. Those bring us to refine this notion to get an accurate computation of the number of maximal subbundles of a sufficiently general parabolic bundle by means of the Intriligator-Varfa formula.

**Key words:** Gromov-Witten invariants, parabolic vector bundles, twisted map, stack.

**MSC2000:** 14B12, 14F05, 14F40, 14H60, 32G08.

## INTRODUCTION

The Gromov-Witten invariants of the Grassmannian that parameterize  $k$ -dimensional vector spaces of the restriction of a vector bundle  $E$  of rank  $r$  and degree  $d$  over a smooth complex projective curve  $C$  of genus  $g(C) = g \geq 2$  over a point  $x$  of  $C$  have been computed [Ho-2] as a an intersection number in the Quot-scheme. These intersection numbers are defined by means of the Schubert schemes that are the degeneracy loci of vector bundles constructed from the property of the universality of the Grassmannian projective scheme and with their correspondence chern class by intersection theory. From this, the Gromov-Witten invariants are defined for a vector bundle  $E$  and weighted homogeneous polynomials  $P$  with extra hypotheses as the  $P(c_1(F^*), \dots, c_k(F^*)) \cap [Quot^{k,e}(E)]$ , where  $Quot$  parameterizes the surjections  $E \rightarrow G$ , with  $G$  is a locally free sheaf of rank  $k$  and degree  $e$  + particular condition on  $e$ . The dual of  $F$ , where  $F$  is defined this time by the universal property of the Quot-scheme. These numbers match with the Gromov-Witten invariants defined in the general framework for the moduli space of stable maps  $f$  from a  $n$ -pointed connected nodal curve  $C$  to the Grassmannian whose  $f_*([C]) = \beta$ .

On the other hand, the number  $m(r, d, k, g)$  is the number of maximal subbundles of a general stable bundle  $E$ , where  $k$  is defined by the  $s_k$ -invariant of  $E$  that is  $s_k(E) = dk - re$ , where  $e = e_{\max, d}(E)$  with  $e = \max\{\deg(F), \text{rk}(F) = k\}$ . This number is none than the number points in the Quot-scheme that is can be simply defined by the weighted homogeneous polynomial.

These numbers are computed by the Intriligator-Varfa formula when the considered morphism is of integer degree.

We are interested in what happens where we readd an extra structure, a parabolic structure for vector bundles (this implies that the morphism can have a rational degree). In this case, the following changes perform as follows. We replace the nodal curve by an orbifoldcurve, the Deligne-Mumford moduli stack of stable maps by the Kontsevich moduli stack of twisted stables maps with their relevant evaluation maps to rigidified inertia stack. Note that the parabolic structure is over marking and node points have a stacky

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structure (root stack structure). In Sect 1, we construct the Quot-stack. In Sect 2, we define the global Gromov-Witten invariants for parabolic case. In Sect 3, those are defined locally by means of the deformation theory. In Sect 4, the latter is defined in a general manner and show that they do not depend on the choice of the orbifold curve. In Sect 5, we establish some properties in the case of general parabolic stable bundles and establish the Intriligator-Varfa formula. In last Sect, we give some examples of computation in the case of finding the number  $m(n, d, k, g)$  of maximal subbundles of a sufficiently parabolic stable bundle.

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## 1. CONSTRUCTION OF THE QUOT-STACK

We start with a twisted curve that is a gadget  $\Sigma_i \subset \mathcal{C} \rightarrow C$ , where  $\mathcal{C}$  is a Deligne-Mumford stack over a coarse moduli connected nodal curve  $C$  such that:

Over a node  $\{xy = 0\}$ , its local chart is  $[\{uv = 0\}/\mu_r]$  given in local coordinates by  $\mathcal{C} \rightarrow C, (u, v) \mapsto (x = u^r, y = v^r)$ .

Over a marking point, its local chart is  $[A^1/\mu_r]$  given in local coordinates by  $\mathcal{C} \rightarrow C, u \mapsto x = u^r$ .

The  $\Sigma_i$  are defined locally by  $\{u = 0\}$ , are gerbes marked by  $\mu_r$ .

**Definition 1.1.** Consider a scheme  $X$ , a line bundle  $\mathcal{L}$  on  $X$ , a section  $s \in \Gamma(X, \mathcal{L})$ , and a positive integer  $r$ . Define a root stack

$$r\sqrt{(L/X, s)},$$

whose objects over a scheme  $Y$  are the triples  $(f : Y \rightarrow X, M, t)$ , where  $M$  is a line bundle on  $Y$  with section  $t$  such that  $M^{\otimes r} \simeq f^*(L)$ ,  $t^r = f^*(s)$ .

For a Cartier divisor  $D$ , we adopt the Vistoli's notation

$$r\sqrt{(X, D)} := r\sqrt{(\mathcal{O}_X(D), \text{id}_D)}.$$

*Remark 1.2.* This stack is isomorphic to  $X$  away from the zero divisor  $D$  of the section, and canonically introduces a stack structure with index  $r$  along  $D$ , is said minimal if  $D$  is smooth. This immediately enables us to define the stacky structure of a twisted curve at marking starting with the coarse moduli curve:

$$(C, p) \hookrightarrow \mathcal{C} = r\sqrt{(C, p)}.$$

To deal with the case of the nodes, we will suppose that the nodes are separating to use root stacks merely, otherwise one needs either subtle descent or logarithmic structures.

We assume that in our case we have a  $n$ -pointed twisted curve  $\mathcal{C}$  and the preimage of  $D$  separated by  $Z \subset C$  the locus of nodes consists of two connected components  $E_1$  and  $E_2$ . Olson established the existence of a universal stack  $\mathcal{C}_{g,n}^{tw}$  of  $n$ -pointed twisted curves of genus  $g$  over the stack of twisted curves  $\mathcal{M}_{g,n}^{tw} = \mathcal{M}$ . We have the structure morphism  $V \rightarrow \mathcal{M}_{g,n}^{tw}$ , where  $V$  is a polydisk. Denote  $\mathcal{M}_r^{tw}$ , the locus where the given node is given stacky structure of index  $r$ , and  $\mathcal{C}_r^{tw}$  the universal twisted curve. Then we have

$$V \times_{\mathcal{M}} \mathcal{M}_r^{tw} = r\sqrt{(V, D)},$$

and

$$V \times_{\mathcal{M}} \mathcal{C}_r^{tw} = r\sqrt{(C, E_1)} \times r\sqrt{(C, E_2)}.$$

Since  $\mathcal{M}_r^{tw} \rightarrow \mathcal{M}$  is birational, but the versal deformation of nodal curves is branched with index  $r$  over  $D$ , this branching is accounted for by automorphisms of the twisted curve. We deduce the automorphism group of a twisted curve fixing  $C$  is

$$Aut(\mathcal{C}) = \prod_{s \in Sing(\mathcal{C})} \Gamma_s,$$

where  $\Gamma_s \simeq \mu_{rs}$  is the stabilizer of the corresponding node.

Let  $\chi$  be a Deligne-Mumford stack over a coarse projective scheme  $X$ . The parabolic structure on a vector bundle over a scheme is parameterized by a product of flag schemes. Moreover we have an equivalence of tensor categories between the category of vector bundles on a stack  $\chi$  and the category of parabolic bundles on a scheme  $X$  after Théorème 3.13 of [Bo-1]. Therefore, it suggests that the parabolic structure is obtained throughout a product of stacks. Hence, we investigate this notion.

We choose  $\chi = Grass(\mathcal{H} = r\sqrt{(C, p_1)} \times \dots r\sqrt{(C, p_n)})$  parameterizes the morphisms of stacks of quotient modules whose objects over a scheme  $T$  of finite type are the cartesian diagrams with relations

$$M_1^{\otimes r} \oplus \dots \oplus M_n^{\otimes r} \simeq f^*(\mathcal{O}_C(p_1) \oplus \dots \oplus \mathcal{O}_C(p_n)), \sum_{i=0}^n t_i^r = \sum_{i=0}^n f^*(s_i).$$

For consequently, there exists a Grassmannian stack of root stacks with a stacky structure at the nodes  $\chi_n = Grass(r\sqrt{(C, E_1)} \times r\sqrt{(C, E_2)})$ . We define the functor

$$\mathbf{Quot}_{e/C}((r\sqrt{(C, p_1)} \times \dots r\sqrt{(C, p_n)}), P) : \mathcal{C}^0 \rightarrow \text{Groupoids}$$

as follows: If  $T$  is a  $k$ -scheme, is associated the groupoid of the cartesian diagrams where  $\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_n$  is a quotient of the direct sum of  $\mathcal{O}_C(p_i)$  flat over  $T$  whose fibers over the geometric points of the  $S$ -Grassmannian projective scheme have Hilbert polynomial  $P_i$  and  $\mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_n$  is a quotient of the direct sum of  $M_i$  whose fibers over the geometric points of the  $S$ -Grassmannian projective scheme have Hilbert polynomial  $P'_i$  with  $P'_i r = P_i$ . The Quot-functor parameterizes the set of  $T$ -flat coherent quotient stacks  $\mathcal{F}$  of  $\mathcal{H}_T$  such that the fiber stacks over the geometric points of the  $S$ -Grassmannian stack have Hilbert polynomial  $P$ .

**Theorem 1.3.** *The functor  $\mathbf{Quot}_{X/S}(\mathcal{H}, P)$  defined above is represented by a projective  $S$ -stack  $\mathbf{Quot}_{X/S}(\mathcal{H}, P)$  with the universal quotient stack  $\mathcal{U}$ .*

*Proof.* See Theorem 2.2.4 of [H-L] in adapting to the case of stacks.  $\square$

In the same way, we also define the functor

$$\mathbf{Quot}_{e/C}(\mathcal{B} = (r\sqrt{(C, E_1)} \times \sqrt{(C, E_2)}), P) : \mathcal{C}^0 \rightarrow \text{Groupoids}$$

is represented by a quasi-projective  $S$ -stack with the universal quotient stack  $\mathcal{U}$ .

We can also consider the stack  $Grass(\mathcal{H})$  as parameterizing the quotient stacks of  $\mathcal{H}$ . Then there is a universal quotient stack  $\oplus \mathcal{K}_i \rightarrow \mathcal{H} \otimes [\mathcal{O}_{\mathcal{H}}]$  with a natural action of the product of the classifying stack  $BGL(\mathcal{H})$ . Let us consider  $H = \oplus H_i$  is contained in the dual of  $\mathcal{H}$ , and  $Y_H$  the Schubert stack defined as the degeneracy locus of  $\oplus H_i \otimes [\mathcal{O}_{\mathcal{H}}] \rightarrow (\oplus K_i)^\wedge$ . Moreover  $Y_H$  decomposes into integral stacks of codimension  $i$ .

**Proposition 1.4.** *The functor*

$$\mathbf{Mor}_e(\mathcal{C}, Grass(\mathcal{H})) : \mathcal{C}^0 \rightarrow \text{Groupoids},$$

is defined as follows: If  $T$  is a scheme over  $S$ , is associated the groupoid of morphisms of degree  $e$  from  $T \otimes \mathcal{C} \rightarrow \text{Grass}(\mathcal{H})$ . The latter is represented by a  $S$  quasi-projective stack denoted  $\text{Mor}_e(\mathcal{C}, \text{Grass}(\mathcal{H}))$ . The functor

$$\text{Mor}_f(\mathcal{C}, \text{Grass}(\mathcal{B})) : \mathcal{C}^0 \rightarrow \text{Groupoids},$$

is defined as follows: If  $T$  is a scheme over  $S$ , is associated the the groupoid of morphisms of degree  $f$  from  $T \otimes \mathcal{C} \rightarrow \text{Grass}(\mathcal{B})$ . The latter is represented by a  $S$ -projective stack denoted  $\text{Mor}_f(\mathcal{C}, \text{Grass}(\mathcal{H}))$ .

*Proof.* We apply Theorem 2.2.4 of [H-L] in adapting to the case of the Grassmannian stack.  $\square$

## 2. GROMOV-WITTEN INVARIANTS

We at present follow the steps of the paper [Ho-2] to define the Gromov-Witten invariants for the parabolic case.

We define the Gromow Witten invariants of the Grassmannian stack as intersection number in the Quot-stack.

**Lemma 2.1.** *Given a product of root stacks, there are some multiintergers  $n = (n_1, \dots, n_n)$  and  $n' = (n_{n+1}, n_{n+2})$  such that for each  $e' = (e - e_1, \dots, e - e_n)$  with  $s'_e \geq n(\mathcal{H})$ , and  $f' = (f - f_{n+1}, f - f_{n+2})$  with  $s'_f \geq n(\mathcal{B})$ , every component of the Quot-stack is generically smooth of expected dimension and a general element in every component corresponds to a substack of the product of root stacks.*

*Proof.* This is proved in [P-R], Theorem 6.4.  $\square$

Let  $(p_1, \dots, p_n, E_1, E_2) \in C^{n+2}$  and substack  $V'_e((p_1, \dots, p_n), \oplus H_i) \subset \text{Quot}^{k, e'}(\mathcal{H})$  defined as the degeneracy locus of

$$\oplus H_i \otimes \mathcal{O}_{\text{Quot}(\mathcal{H})} \rightarrow (pr_1(\mathcal{H}), \dots, pr_n(\mathcal{H})) \rightarrow ((F_1)^\wedge \oplus \dots \oplus (F_n)^\wedge).$$

We define in the same way in the case of the nodes for a substack  $V'_f((E_1, E_2), \oplus H'_i) \subset \text{Quot}^{k, f'}(\mathcal{B})$ .

Denote  $s'_e = (dk - re') = (d_1 k_1 - ne_1, \dots, d_n k_n - ne_n)$  and  $s'_f = (d_{n+1} k_{n+1} - 2f_{n+1}, d_{n+2} k_{n+2} - 2f_{n+2})$ . Let  $X_1, \dots, X_k$  be weighted variables such that the weight of  $X_i$  is  $i$ . Let  $P(X_1, \dots, X_{k_n})$  be a weighted homogeneous polynomial of weighted degree  $s'_e + k_n(n - k_n)(1 - g)$  (resp.  $s'_f + k_{n+2}(2 - k_{n+2})(1 - g)$ ) with  $s'_e > s(\mathcal{H})$  (resp.  $s'_f > s(\mathcal{B})$ ).

**Definition 2.2.** For a pair  $(\mathcal{H}, e')$  over  $C$  and  $(\mathcal{B}, f')$ , we define the Gromov-Witten invariants  $N_{n, e'} P(X_1, \dots, X_{k_n})$  as

$$N_{n, e'} P(X_1, \dots, X_{k_n}) = P\left(\prod_{i=1}^n (c_i((F_i)^\wedge), \dots, c_{k_i}((F_i)^\wedge)) \cap [\text{Quot}_{vir}^{(k_n, e')}(\mathcal{H})],\right.$$

and

$$N_{n', f'} P(X_1, \dots, X_{k_{n+2}}) = P\left(\prod_{i=1}^2 (c_{k_{n+i}}((F_{n+i})^\wedge), c_{k_{n+i+1}}((F_{n+i})^\wedge)) \cap [\text{Quot}_{vir}^{(k_{n+2}, f')}(\mathcal{B})].\right.$$

We note that our  $\text{Quot}_{X/S}(r\sqrt{(C, p_1)} \times \dots \times r\sqrt{(C, p_n)})$  can be viewed as an augmented simplicial stack. We now consider a product of such simplicial stacks which are homotopic up to permutation to give a parameterization for parabolic bundles. We apply the cohomology descent so that we may recover a product of flag schemes that

parameterizes the parabolic structure, and for consequently get the parabolic bundles on a scheme  $X$ . We just argue this in specifying that our parabolic bundle is a functor  $E_* : (\frac{1}{r}\mathbb{Z})^{op} \rightarrow \mathbf{Vect}(X)$  whose its degree is given by Théorème 4.3 [Bo-1]

$$\text{par deg } E_* = \deg_X \nu = q_*(c_1^{et}(\nu)).\pi^* c_1^{et} \mathcal{O}_X(1)^{n-1},$$

where  $q : X \rightarrow \text{Spec}(\mathbb{C})$ , and apply the equivalence between tensor categories so that the Gromov-Witten invariants may be defined as above.

We at present want to see the relation between the Gromov-Witten invariants defined above and the one known in the case of the Kontsevich moduli stack  $\mathcal{K}_{g,n}(\chi, \beta)$  of stable twisted maps.

**Proposition 2.3.**

$$\mathcal{K}_{g,n}(\text{Grass}(\mathcal{H}, \beta_e)) \simeq \text{Quot}(\mathcal{H}) \times \mathbb{C}^n,$$

and

$$\mathcal{K}_{g,n}(\text{Grass}(\mathcal{B}, \beta_f)) \simeq \text{Quot}(\mathcal{B}) \times \mathbb{C}^2.$$

*Proof.* Straightforward. □

### 3. CASE OF DEFORMATION THEORY

We want at present to construct the Gromov-Witten invariants for parabolic vector bundles over an orbifold locally in using deformation theory.

We first construct the versal deformation of  $(t, \lambda)$  parabolic connections to deduce the Kuranishi space of parabolic bundles.

We set

$$T_n := \left\{ (t_1, \dots, t_n) \in \overbrace{X \times \dots \times X}^n \mid t_i \neq t_j \text{ for } i \neq j \right\}$$

for a positive integer  $n$ . For integers  $d, r$  with  $r > 0$ , we set

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r-1}} \in \mathbb{C}^{nr} \mid d + \sum_{i,j} \lambda_j^{(i)} = 0 \right\}.$$

Take an element  $t = (t_1, \dots, t_n) \in T_n$  and  $\lambda = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \Lambda_r^{(n)}(d)$ .

**Definition 3.1.**  $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$  is said to be a  $(t, \lambda)$ -parabolic connection of rank  $r$  if

- (1)  $E$  is a rank  $r$  algebraic vector bundle on  $X$ , and
- (2)  $\nabla : E \rightarrow E \otimes \Omega_C^1(\log(t_1 + \dots + t_n))$  is a connection, and
- (3) for each  $t_i$ ,  $l_*^{(i)}$  is a filtration of  $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$  such that  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$  and  $(\text{Res}_{t_i}(\nabla) - \lambda_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $j = 0, \dots, r-1$ .

*Remark 3.2.* By condition (3) above and [EV-1], we have

$$\deg E = \deg(\det(E)) = - \sum_{i=1}^n \text{Tr Res}_{t_i}(\nabla) = - \sum_{i=1}^n \sum_{j=0}^{r-1} \lambda_j^{(i)} = d.$$

Let  $T$  be a smooth algebraic scheme which is a covering of the moduli stack of  $n$ -pointed smooth projective curves of genus  $g$  over  $\mathbb{C}$  and take a universal family  $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$  over  $T$ .

**Definition 3.3.** We denote the pull-back of  $\mathcal{C}$  and  $\tilde{t}$  with respect to the morphism  $T \times \Lambda_r^{(n)}(d) \rightarrow T$  by the same characters  $\mathcal{C}$  and  $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_n)$ . Then  $D(\tilde{t}) := \tilde{t}_1 + \dots + \tilde{t}_n$  becomes a family of Cartier divisors on  $\mathcal{C}$  flat over  $T \times \Lambda_r^{(n)}(d)$ . We also denote by  $\tilde{\lambda}$  the pull-back of the universal family on  $\Lambda_r^{(n)}(d)$  by the morphism  $T \times \Lambda_r^{(n)}(d) \rightarrow \Lambda_r^{(n)}(d)$ . We define a functor  $M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d)$  from the category of locally noetherian schemes over  $T \times \Lambda_r^{(n)}(d)$  to the category of sets by

$$M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

where

- (1)  $E$  is a vector bundle on  $\mathcal{C}_S = \mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} S$  of rank  $r$ ,
- (2)  $\nabla : E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1(D(\tilde{t})_S)$  is a relative connection,
- (3)  $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$  is a filtration by subbundles such that  $(\text{Res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}_j^{(i)})_S)(l_j^{(i)}) \subset l_{j+1}^{(i)}$  for  $0 \leq j \leq r-1, i = 1, \dots, n$ ,
- (4) for any geometric point  $s \in S$ ,  $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$  for any  $i, j$  and  $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$  is  $\alpha$ -stable.

Here  $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l_j'^{(i)}\})$  if there exist a line bundle  $\mathcal{L}$  on  $S$  and an isomorphism  $\sigma : E \xrightarrow{\sim} E' \otimes \mathcal{L}$  such that  $\sigma|_{l_i}(l_j^{(i)}) = l_j'^{(i)}$  for any  $i, j$  and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{t})) \\ \sigma \downarrow & & \sigma \otimes \text{id} \downarrow \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla'} & E' \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{t})) \otimes \mathcal{L} \end{array}$$

commutes.

We now can construct the moduli space of this functor.

**Theorem 3.4.** *There exists a relative fine moduli scheme*

$$M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d) \rightarrow T \times \Lambda_r^{(n)}(d)$$

of  $\alpha$ -stable parabolic connections of rank  $r$  and degree  $d$ , which is smooth, irreducible and quasi-projective and has an algebraic symplectic structure. The fiber  $M_{\mathcal{C}_x}^\alpha(\tilde{t}_x, \lambda)$  over  $(x, \lambda) \in T \times \Lambda_r^{(n)}(d)$  is the irreducible moduli space of  $\alpha$ -stable  $(\tilde{t}_x, \lambda)$  parabolic connections whose dimension is  $2r^2(g-1) + nr(r-1) + 2$  if it is non-empty.

*Proof.* See [I]. □

Let  $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$  be a universal family on  $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d)$ . We define a complex  $\mathcal{G}^\bullet$  by

$$\begin{aligned} \mathcal{G}^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \mid s|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{G}^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{t})) \mid \text{Res}_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{t}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\} \\ \nabla_{\mathcal{G}^\bullet} : \mathcal{G}^0 &\longrightarrow \mathcal{G}^1; \quad \nabla_{\mathcal{G}^\bullet}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}. \end{aligned}$$

As in the previous section, we can construct the Kuranishi space of  $(t, \lambda)$ -parabolic connections on a smooth projective curve in using the hypercohomology of  $\mathcal{G}^\bullet$ .

**Theorem 3.5.** Let  $X$  be a smooth projective curve over  $k$ ,  $(\mathcal{E}, \nabla, \{l_*^{(i)}\})$  a  $(t, \lambda)$ -parabolic connection on  $X$ ,  $\mathcal{G}^\bullet$  the complex of sheaves on  $X$  defined above,  $W = \mathbb{H}^1(X, \mathcal{G}^\bullet)$ ,  $(\delta_1, \dots, \delta_N)$  a basis of  $W$  and  $(t_1, \dots, t_N)$  the dual coordinates on  $W$ . Let  $W_k$  denote the  $k$ -th infinitesimal neighborhood of 0 in  $W$ , and  $(\mathcal{E}_1, \nabla_1, \{l_*^{(i)}\}_1)$  the universal first order deformation of  $(\mathcal{E}, \nabla, \{l_*^{(i)}\})$  over  $X \times W_1$  in the class of  $(t, \lambda)$ -parabolic connections. Then there exists a formal power series

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1, \dots, t_N) \in \mathbb{H}^2(X, \mathcal{G}^\bullet)[[t_1, \dots, t_N]],$$

where  $f_k$  is homogeneous of degree  $k$  ( $k \geq 2$ ), with the following property. Let  $I$  be the ideal of  $k[[t_1, \dots, t_N]]$  generated by the image of the map  $f^* : \mathbb{H}^2(X, \mathcal{G}^\bullet) \rightarrow k[[t_1, \dots, t_N]]$ , adjoint to  $f$ . Then for any  $k \geq 2$ , the triple  $(\mathcal{E}_1, \nabla_1, \{l_*^{(i)}\}_1)$  extends to a  $(t, \lambda)$ -parabolic connection  $(\mathcal{E}_k, \nabla_k, \{l_*^{(i)}\}_k)$  on  $X \times V_k$ , where  $V_k$  is the closed subscheme of  $W_k$  defined by the ideal  $I \otimes k[[t_1, \dots, t_N]] / (t_1, \dots, t_N)^{k+1}$ .

*Proof.* This follows of the proof by construction in Theorem 3.6 of [Machu].  $\square$

We now want to construct the Kuranishi space of  $T$ -parabolic bundles. Let  $T$  be a finite set of smooth points  $\{P_1, \dots, P_n\}$  of  $X$  and  $W$  a vector bundle on  $X$ .

**Definition 3.6.** By a quasi-parabolic structure on a vector bundle  $W$  at a smooth point  $P$  of  $X$ , we mean a choice of a flag

$$W_P = F_1(W)_P \supset F_2(W)_P \supset \dots \supset F_l(W)_P = 0,$$

in the fibre  $W_P$  of  $W$  at  $P$ . A parabolic structure at  $P$  is a pair consisting of a flag as above and a sequence  $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$  of weights of  $W$  at  $P$ .

The integers  $k_1 = \dim F_1(W)_P - \dim F_2(W)_P, \dots, k_l = \dim(F_l(W)_P)$  are called the multiplicities of  $\alpha_1, \dots, \alpha_l$ . A  $T$ -parabolic structure on  $W$  is the triple consisting of a flag at  $P$ , some weights  $\alpha_i$ , and their multiplicities  $k_i$ . A vector bundle  $W$  endowed with a  $T$ -parabolic structure is called a  $T$ -parabolic bundle.

**Definition 3.7.** A  $T$ -parabolic bundle  $W_1$  on  $X$  is a  $T$ -parabolic subbundle of a  $T$ -parabolic bundle  $W_2$  on  $X$ , if  $W_1$  is a subbundle of  $W_2$  and at each smooth point  $P$  of  $T$ , the weights of  $W_1$  are a subset of those of  $W_2$ . Further, if we take the weight  $\alpha_{j_0}$  such that  $1 \leq j_0 \leq m$ , and the weight  $\beta_{k_0}$  for the greatest integer  $k_0$  such that  $F_{j_0}(W_1)_P \subset F_{k_0}(W_2)_P$ , then  $\alpha_{j_0} = \beta_{k_0}$ .

**Definition 3.8.** The parabolic degree of a  $T$ -parabolic vector bundle  $W$  on  $X$  is

$$\text{par deg}(W) := \deg(W) + \sum_{P \in T} \sum_{i=1}^r k_i(P) \alpha_i(P).$$

**Definition 3.9.** A  $T$ -parabolic bundle  $W$  is stable (resp. semistable) if for any proper nonzero  $T$ -parabolic subbundle  $W' \subset W$  the inequality

$$\text{par deg } W' < (\text{resp. } \leq) \frac{\text{par deg } W \cdot \text{rk}(W')}{\text{rk } W}$$

holds.

We have a forgetful map  $g$  from  $(t, \lambda)$  parabolic connections to  $T$ -parabolic bundles. We thus can construct the Kuranishi space of  $T$ -parabolic bundles by following an analogous argument to the one given above. We first introduce the Higgs field  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1(D)$  defined as follows:

$$\forall p \in X, \forall f \in \mathcal{O}_{X,p}, \forall s \in \mathcal{E}_p, \Phi(fs) = f\Phi(s).$$

We afterwards consider a parabolic bundle  $\mathcal{E}$  with fixed weights and parabolic points  $P_1, \dots, P_N$ . We set  $L = K \otimes \mathcal{O}(P_1, \dots, P_N)$ , the line bundle associated to the canonical divisor together with the divisor of poles  $D = P_1 + \dots + P_N$ . The sheaf of rational 1-forms on  $X$  is identified with the sheaf of rational sections of the canonical bundle having single poles at points  $P_1, \dots, P_N$ . We replace  $t_i$  by  $P_i$ , for  $i = 1, \dots, N$  and  $M_{\mathbb{C}/T}^\alpha(\tilde{t}, r, d)$  by  $M_T^s$ . We define a complex  $\mathcal{B}^\bullet$  by

$$\begin{aligned} \mathcal{B}^0 &:= \left\{ s \in \text{End}(\tilde{E}) \mid s|_{\tilde{P}_i \times M_{Z, \mathbb{C}/T}^s(\tilde{P}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{B}^1 &:= \left\{ s \in \text{End}(\tilde{E}) \otimes \Omega_{\mathbb{C}/T}^1(D(\tilde{P})) \mid \text{Res}_{\tilde{P}_i \times M_{Z, \mathbb{C}/T}^s(\tilde{P}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\} \end{aligned}$$

$$\text{ad } \Phi_{\mathcal{B}^\bullet} : \mathcal{B}^0 \longrightarrow \mathcal{B}^1; \quad \text{ad } \Phi_{\mathcal{B}^\bullet}(s) = \tilde{\Phi} \circ s - s \circ \tilde{\Phi}.$$

From this, we deduce the construction of the Kuranishi space of  $T$ -parabolic bundles on a smooth projective curve.

**Theorem 3.10.** *Let  $X$  be a smooth projective curve over  $k$  or a complex space (in which case  $k = \mathbb{C}$ ),  $\mathcal{E}$  a  $T$ -parabolic bundle on  $X$ ,  $\mathcal{B}^\bullet$  the complex of sheaves on  $X$  defined as above,  $W = \mathbb{H}^1(X, \mathcal{B}^\bullet)$ ,  $(\delta_1, \dots, \delta_N)$  a basis of  $W$  and  $(t_1, \dots, t_N)$  the dual coordinates on  $W$ . Let  $W_k$  denote the  $k$ -th infinitesimal neighborhood of 0 in  $W$ , and  $\mathcal{E}_1$  the universal first order deformation of  $\mathcal{E}$  over  $X \times W_1$ . Then there exists a formal power series*

$$f(t_1, \dots, t_N) = \sum_{k=2}^{\infty} f_k(t_1, \dots, t_N) \in \mathbb{H}^2(X, \mathcal{B}^\bullet)[[t_1, \dots, t_N]],$$

where  $f_k$  is homogeneous of degree  $k$  ( $k \geq 2$ ), with the following property. Let  $I$  be the ideal of  $k[[t_1, \dots, t_N]]$  generated by the image of the map  $f^* : \mathbb{H}^2(X, \mathcal{B}^\bullet)^* \rightarrow k[[t_1, \dots, t_N]]$ , adjoint to  $f$ . Then for any  $k \geq 2$ ,  $\mathcal{E}_1$  extends to a  $T$ -parabolic bundle  $\mathcal{E}_k$  on  $X \times V_k$ , where  $V_k$  is the closed subscheme of  $W_k$  defined by the ideal  $I \otimes k[[t_1, \dots, t_N]]/(t_1, \dots, t_N)^{k+1}$ .

*Proof.* This follows of the proof by construction in Theorem 3.6 of [Machu].  $\square$

**Definition 3.11.** The inverse limit  $\mathbb{V} = \varprojlim V_k$  is called the formal Kuranishi space of  $\mathcal{E}$ , and  $\mathcal{E} = \varprojlim \mathcal{E}_k$  the formal universal parabolic bundle over  $\mathbb{V}$ .

We can hence apply the previous method of constructing locally the Gromov-Witten invariants of parabolic bundles over an orbifold.

**Definition 3.12.** For a pair  $(\mathcal{H}, e')$  over  $C$  and  $(\mathcal{B}, f')$ , we define the Gromov-Witten invariants  $N_{n, e'} P(X_1, \dots, X_{k_n})$  as

$$N_{n, e'} P(X_1, \dots, X_{k_n}) = P\left(\prod_{i=1}^n (c_i((\mathbf{F}_i)^\wedge), \dots, c_{k_i}((\mathbf{F}_i)^\wedge)) \cap [Quot_{vir}^{(k, e')}(\mathcal{H})],\right.$$



and

$$N_{n',f'}P(X_1, \dots, X_{k_{n+2}}) = P\left(\prod_{i=1}^2 (c_{k_{n+i}}((\mathbf{F}_{n+i})^\wedge, c_{k_{n+i+1}}((\mathbf{F}_{n+i})^\wedge)) \cap [Quot_{vir}^{(k,f')}(\mathcal{B})]\right).$$

#### 4. GENERALIZATION

It is then natural to ask what happens in the framework of the generalization of the primitive definition of a parabolic structure at a marked point.

We at present consider a connected complex reductive algebraic group  $G$  containing a simply-connected and simple compact group  $K$  such  $T$  its maximal torus in  $K$  and  $P$  a parabolic subgroup of  $G$ . We denote  $W$  the Weyl group and  $W_P$  its subgroup generated by the simple reflection of roots of the Levi subgroup of  $P$ . We also denote  $\mathfrak{t}$  the cartan subalgebra containing  $\mathfrak{t}_+$  the positive Weyl chamber and  $\alpha_0$  the highest root. Let  $\pi : E \rightarrow C$  be a principal  $G$ -bundle over  $C$  with marked points  $p_1, \dots, p_n$ .

**Definition 4.1.** A parabolic structure at  $p_i$  consists of the following data:

- (1) a standard parabolic subgroup  $P_i \subset G$ .
- (2)  $\varphi_i \in E_x/P_i$  of the reduction of the fiber  $E_x$  to  $P_i$ .
- (3) a marking  $\mu_i \in \mathcal{U}$ , where  $\mathcal{U} = \{\epsilon \in \mathfrak{t}_+ \mid \alpha_0(\epsilon) \leq 1\}$  with  $\alpha_0(\mu_i) < 1$ , where the stabilizer  $G_{\mu_i}$  under the adjoint action is a Levi subgroup of  $P_i$ .

Hence a parabolic bundle on  $(C, p_1, \dots, p_n)$  is a bundle  $E$  with parabolic structure at these points.

**Definition 4.2.** A reduction of structure group of  $E$  at  $P$  is a map

$$\sigma : C \rightarrow E/P.$$

Note that for any  $\lambda \in \Lambda_P$ , where  $\Lambda_P$  is the character subgroup of  $P$ ,  $\sigma^*(E(\lambda))$  is a line bundle on  $C$  whose degree is in  $\mathbb{Z}$ . The latter will be used to define the root stacks as previously.

One of the goal of this paper is to determine the number of maximal parabolic subbundles of a sufficiently general stable bundle. So, we make reference to the definition of Ramanathan for the semistability.

**Definition 4.3.**  $E$  semistable if  $\deg(\sigma^*(E(\lambda))) \leq 0, \forall \lambda \in \Lambda_{P,+}$ .

Hence, we see that the definition of semistability for parabolic principal  $G$ -bundles depend on the relative position of  $\sigma$  and  $\varphi$ . Given two parabolic subgroups  $P'_1 = \text{Ad}(g)P_1$ ,  $P'_2 = \text{Ad}(g)P_2$ , define their relative position  $(P'_1, P'_2) \in WP_1 \setminus W/WP_2$  to be the image of  $(g_1, g_2)$  under the map

$$G \times G \rightarrow WP_1 \setminus W/WP_2.$$

We deduce the following definition.

**Definition 4.4.**  $E$  is stable (resp. semistable) if for any maximal subgroup  $P$  of  $G$  and  $\sigma$ , we have

$$\deg(\sigma^*(E(\lambda))) + \sum_{i=1}^p \omega_P(w_i \mu_i) < 0 (\text{resp. } \leq 0),$$

where  $w_i = (\varphi_i, \sigma(p_i))$  and  $\omega_p$  fundamental weights of  $P$ .

We apply our previous results to this case and get the definition of the Gromov-Witten invariants in the case of parabolic bundles where we replace  $\mathcal{H} = r\sqrt{(C, p_1)} \times \dots r\sqrt{(C, p_n)}$  by  $\mathcal{H} = r\sqrt{(\sigma^*(E(\lambda)), p_1)} \times \dots r\sqrt{(\sigma^*(E(\lambda)), p_n)}$ , idem for  $\mathcal{B} = (r\sqrt{(C, E_1)} \times \sqrt{(C, E_2)})$ . Finally, we arrive at the following definition:

**Definition 4.5.** For a pair  $(\mathcal{H}, e')$  over  $C$  and  $(\mathcal{B}, f')$ , we define the Gromov-Witten invariants  $N_{n, e'} P(X_1, \dots, X_{k_n})$  as

$$N_{n, e'} P(X_1, \dots, X_{k_n}) = P\left(\prod_{i=1}^n (c_i((F_i)^\wedge), \dots, c_{k_i}((F_i)^\wedge)) \cap [\text{Quot}_{\text{vir}}^{(k, e')}[\mathcal{H}]]\right),$$

and

$$N_{n', f'} P(X_1, \dots, X_{k_{n+2}}) = P\left(\prod_{i=1}^2 (c_{k_{n+i}}((F_{n+i})^\wedge), c_{k_{n+i+1}}((F_{n+i})^\wedge)) \cap [\text{Quot}_{\text{vir}}^{(k, f')}[\mathcal{B}]]\right).$$

We show that the Gromov-Witten invariants defined hence are independent on the choice of the orbifold curve  $C$  of genus  $g$ . For this, we first note that for a collection of root stacks over a genus- $g$  curve, the existence of a smooth irreducible variety  $T$  and a family  $\mathcal{F}$  of root stacks on  $C \times T$  whose restriction at the fiber over a point in  $T$  gives a root stack, constructed in using the universal property arising from the smoothness and the irreducibility of the moduli stack of roots. Therefore, we obtain for a such family on  $C \times T$ .

**Proposition 4.6.** *If  $\mathcal{F}$  is a family of root stacks on  $C \times T$ , with  $T$  a smooth curve, and  $e$  being chosen such that  $s_e \geq s_{\mathcal{F}}$ , then the Gromov-Witten invariants are independent of the choice of points  $x \in B$ .*

*Proof.* We show that the relative Quot-scheme  $e : \text{Quot}(\mathcal{F}) \rightarrow T$  is a locally complete intersection morphism, and in particular flat. For this, the hypothesis  $s_e \geq s_{\mathcal{F}}$  enables to say that  $\text{Quot}(\mathcal{F}_x)$  is generically smooth of expected dimension. Therefore, the proposition follows from Lemma 1.6 of [Ber].  $\square$

**Lemma 4.7.** *Let  $\mathcal{H}$  be the root stack of multidegree  $e$  and of rank  $n$  and  $L$  a line bundle of degree  $d$ . Then we have  $s(\mathcal{H}) = s(\mathcal{H} \otimes L)$ , and the Gromov Witten invariants of  $\mathcal{H}$  and  $\mathcal{H} \otimes L$  are related by the following formula*

$$N_{e'+nd, e'+k_nd}(P(X_1, \dots, X_{k_n}), \mathcal{H} \otimes L) = N_{d, e'}(P(X_1, \dots, X_{k_n}), \mathcal{H}).$$

*Proof.* This follows arising immediately from the isomorphism between the Quot stacks  $\text{Quot}^{k_n, e'+k_nd}(\mathcal{H} \otimes L)$  and  $\text{Quot}^{k_n, e'}(\mathcal{H})$ .  $\square$

Before continuing our study in the case of general stable parabolic bundles, we make an another approach to refine the construction of the Gromov-Witten invariants for parabolic bundles in using the correspondence between the equivariant bundles and the parabolic bundles.

We start with a cyclic group  $\Gamma$  of order  $N$  acting on a curve  $\tilde{X}$  with quotient  $X = \tilde{X}/\Gamma$  with a map  $\pi : \tilde{X} \rightarrow X$ , ramified at the  $x_i$ . Fix  $\mu_1, \dots, \mu_n \in \mathcal{U}$  with  $e^{\mu_i} = 1, \forall 1 \leq i \leq n$ . Mehta-Seshadri proved that there is a one-to-one correspondence between the set of isomorphism classes of  $\Gamma$ -equivariant bundles  $\tilde{E}$  on  $\tilde{X} \times S$  with a  $\Gamma$ -action on the fibers  $E_{x_i}$  lie in the conjugation class of  $e^{u_i}$  with the set of isomorphism classes of parabolic bundles  $E$  on  $X \times S$ .

It is not difficult to pass from the construction of an equivariant bundle to a parabolic bundle and conversely where the parabolic structure is given by the filtration at the ramified points by order of the vanishing. To recover the parabolic bundle, we quotient by the group  $\Gamma$  and use the transition functions  $z^{-N\mu_i/2i\pi}$ .

We apply our previous results to this case and get the definition of the Gromov-Witten invariants in the case of parabolic bundles. Finally, we arrive at the following definition:

**Definition 4.8.** For a pair  $(\mathcal{H}, e')$  over  $C$  and  $(\mathcal{B}, f')$ , we define the Gromov-Witten invariants  $N_{n,e'}P(X_1, \dots, X_{k_n})$  as

$$N_{n,e'}P(X_1, \dots, X_{k_n}) = P\left(\prod_{i=1}^n (c_i((F_i)^{-N\mu_i}/\Gamma)^\wedge, \dots, (c_{k_i}((F_i)^{-N\mu_i}/\Gamma)^\wedge)) \cap [Quot_{vir}^{(k,e')}[\mathcal{H}]],\right.$$

and

$$N_{n',f'}P(X_1, \dots, X_{k_{n+2}}) = P\left(\prod_{i=1}^2 (c_{k_{n+i}}((F_{n+i})^{-N\mu_i}/\Gamma)^\wedge, c_{k_{n+i+1}}((F_{n+i+1})^{-N\mu_i}/\Gamma)^\wedge) \cap [Quot_{vir}^{(k,f')}[\mathcal{B}]].\right.$$

## 5. GENERAL STABLE PARABOLIC BUNDLES

We now assume that we work with the moduli stacks  $Quot^{(k,e')}[\mathcal{H}]$  and  $[Quot^{(k,f')}[\mathcal{B}]]$  of stable objects. In the case of the stability of the objects, we can define the notion of  $s$ -invariant as follows.

$$s_{k_n}(\mathcal{H}) = k_n(n - k_n)(g - 1) + \epsilon, 1 \leq \epsilon \leq n - 1, s_{k_{n+2}}(\mathcal{B}) = k_{n+2}(2 - k_{n+2})(g - 1) + 1.$$

Let  $e_{max,d}$  be the degree of the maximal subbundle of a general stable bundle of degree  $d$ . We can also define those in the refinement of our definition for the Gromov-Witten invariants.

$$s_{k_n}(\mathcal{H}) = k_n(n - k_n)(g - 1) + \epsilon + N \sum_{i=1}^n \mu_i, 1 \leq \epsilon \leq n - 1, s_{k_{n+2}}(\mathcal{B}) = k_{n+2}(2 - k_{n+2})(g - 1) + 1 + N \sum_{i=1}^2 \mu_i.$$

**Proposition 5.1.** *The moduli stack of roots admits an open moduli stack  $\mathcal{U}$  with the property that for each  $H \in \mathcal{U}$ , and for each  $e' \leq e_{max,n}$  (resp.  $e' \leq e_{max,2}$ , every component of the quot stack  $Quot^{k_n,e'}(H)$  (resp.  $Quot^{k_{n+2},f'}(H)$ ) is smooth of expected dimension and satisfies the property that general elements in every irreducible component correspond to root substacks of  $H$ .*

*Proof.* The proof of the Proposition relies on the torsion free part of the Quot-stack is generically smooth from Proposition 6.7 of [Ho-1] and on the contradiction on the dimensions show that an irreducible component of the Quot-stack is torsion free  $\square$

We search for some relations between the Gromov Witten invariants for parabolic bundles.

**Theorem 5.2.** *Let  $n$  and a multiinteger  $d$  be fixed. Set  $d = ar - b$ , where  $0 \leq b < n$  and  $e \leq e_{max}(d)$ . Let  $P(X_1, \dots, X_{k_n})$  be a polynomial of weighted degree*

$$d_1 k_1 - n e_1 + k_1(n - k_1)(1 - g), \dots, d_n k_n - n e_n + k_n(n - k_n)(1 - g).$$

*Then we deduce the following relation*

$$N_{d,e'}(P(X_1, \dots, X_{k_n})) = \prod_{a,b,k} N_{0,e'-ak_n}(X_{k_n}^b P(X_1, \dots, X_{k_n})).$$

*Proof.* This follows of the previous Proposition and Lemma 4.7.  $\square$

We can also note that this shows that this is independent of the choice of the orbifold curve  $C$  of genus  $g$ . We at present remind the formula of Vafa and Intriligator, proved by A. Bertram (see in [Ber], [Ber-Das-Went] updated to our case for an explicit computation of Gromov Witten invariants  $N_{0,e'}(P(X_1, \dots, X_{k_n}))$ . Let  $P(X_1, \dots, X_{k_n}) = \prod_{i=1}^m X_{a_i}$  be a polynomial with  $0 < a_i \leq k_n$  such that the weighted degree of  $P$  is  $\sum_i (k_n - a_i + 1) = -e'n + k_n(n - k_n)(1 - g)$ . Then we have the following.

**Proposition 5.3.** *For the polynomial  $P = \prod_{i=1}^m X_{a_i}$ , defined as above, the Gromov Witten invariants are constructed as follows. We introduce a few notation  $k' = k_n$ ,  $\alpha = k'(g - 1)$ ,  $\beta = (-1)^{e'(k'-1)+(g-1)k'(k'-1)/2}$ , and  $S = \{(\rho_1, \dots, \rho_{k'}) \mid \rho_i^n = 1, \rho_i \neq \rho_j\}$  and  $\Delta = \prod_{i=1}^m \sigma_{k'-a(l)+1}(\rho)$  to get*

$$\frac{n^{\alpha\beta}}{k'!} \sum_S \frac{\Delta}{(\prod_{i=1}^n \rho_i \prod_{i \neq j} (\rho_i - \rho_j))^{g-1}},$$

where  $\sigma_j(\rho)$  is the  $j$ -symmetric polynomial in  $\rho_i$ 's.

## 6. MAXIMAL PARABOLIC SUBBUNDLES

We want to provide some examples of the computation of our Gromov-Witten invariants for parabolic bundles over an orbifold  $C$  of genus  $g$ , in particular the number of maximal subbundles of a sufficiently parabolic stable bundle denoted  $m(n, d, k_n, g)$  and  $m(2, d, k_{n+2}, g)$  in certain cases. We first state the following proposition.

**Proposition 6.1.** *For a general root stack  $\mathcal{E}$ , the Quot-stack  $\text{Quot}^{k_n, e'_{\max, d}}(\mathcal{E})$  is a zero-dimensional smooth stack.*

*Proof.* We are in the case where  $e' = e'_{\max, d}$ , hence in using the result of Mukai and Sakai [M-S], with Lemma 4.7, we deduce the result.  $\square$

Futhermore, we can count the number of points  $m(n, d, k_n, g)$  (resp.  $m(2, d, k_{n+2}, g)$ ) in the Quot stack.

**Theorem 6.2.** *In using Theorem 5.2, we get with  $\beta = (-1)^{(k'-1)(bk'-(g-1)k'^2/n)}$ ,*

$$\frac{n^{\alpha\beta}}{k'!} \sum_S \frac{\Delta^{b-g+1}}{(\prod_{i \neq j} (\rho_i - \rho_j))^{g-1}}.$$

*Proof.* Use Proposition 5.3 and Theorem 5.2.  $\square$

We deduce the following Corollary.

**Corollary 6.3.**  $m(n, d, 1, g) = n^{ng}$ , and  $m(2, d, 1, g) = n^{2g}$ .

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